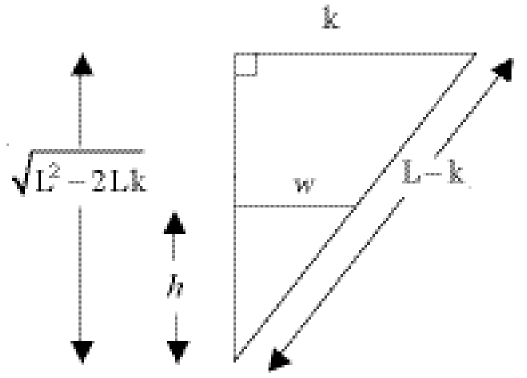
NEW ZEALAND SCHOLARSHIP 2004

ASSESSMENT SCHEDULE FOR CALCULUS

No.	Evidence	Code	Judgement
1(a)	Slant height of the pyramid = $\frac{2\text{L}-2x}{2} = \text{L}-x$ By Pythagoras' theorem $\begin{aligned}(\text{height of pyramid})^2 &= (\text{L}-x)^2 - x^2 \\ &= \text{L}^2 - 2\text{L}x + x^2 - x^2 \\ &= \text{L}^2 - 2\text{L}x\end{aligned}$ height of pyramid $=\sqrt{\text{L}^2 - 2\text{L}x}$ Volume $= \frac{1}{3}(2x)^2 \times \sqrt{\text{L}^2 - 2\text{L}x}$ $= \frac{1}{3}4x^2 \times \sqrt{\text{L}^2 - 2\text{L}x}$ cm^3	BM	Units of measurement not required. Any correct alternative method acceptable (eg use of different triangles).
	Hence the value of x to give max. volume: $\frac{dV}{dx} = \frac{4}{3}2x\sqrt{\text{L}^2 - 2\text{L}x} + \frac{4}{3}x^2\frac{1}{2}(\text{L}^2 - 2\text{L}x)^{\frac{-1}{2}}(-2\text{L})$	C	Accept maximising V^2
	$= \frac{4x}{3}(\text{L}^2 - 2\text{L}x)^{\frac{-1}{2}}\left[2(\text{L}^2 - 2\text{L}x) - \text{L}x\right]$ $= \frac{4x}{3}(\text{L}^2 - 2\text{L}x)^{\frac{-1}{2}}\left[2\text{L}^2 - 5\text{L}x\right] = 0$ for max. & min. $x = 0$ or $x = \frac{2\text{L}^2}{5\text{L}} = \frac{2\text{L}}{5}$ $(\text{L} \neq 0)$ So for maximum volume $x = \frac{2\text{L}}{5}$ cm		$x = 0$ not required. Units not required No alternative.

No.	Evidence	Code	Judgement
1(b)	Given $\frac{dV}{dt}=\frac{k^2}{3L}$; to find $\frac{dh}{dt}$.  Using similar triangles: $\frac{w}{k}=\frac{h}{\sqrt{L^2-2Lk}}$ $w=\frac{kh}{\sqrt{L^2-2Lk}}$ $V=\frac{1}{3}(2w)^2h$ $=\frac{4}{3}\times\frac{k^2h^3}{L^2-2Lk}$ Using the Chain Rule: $\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}$ $\frac{dV}{dt}=\frac{4k^2h^2}{L(L-2k)}\times\frac{dh}{dt}$ $\frac{dh}{dt}=\frac{L(L-2k)}{4k^2h^2}\times\frac{k^2}{3L}=\frac{L-2k}{12h^2}$ Hence, when $\frac{dh}{dt}>\frac{L-2k}{h+1}$ $\frac{L-2k}{12h^2}>\frac{L-2k}{h+1}$ $h+1>12h^2 \quad (h\ge 0,\ L-2k\ne 0)$ $\therefore 12h^2-h-1<0$ $(4h+1)(3h-1)<0$ $\therefore 0\le h<\frac{1}{3}$ cm. $(h\ge 0)$	**AR** (Possible replacement for AP).	Any correct alternative method acceptable. Or equivalent: eg $\frac{dh}{dt}=\frac{dh}{dV}\times\frac{dV}{dt}$ Units of measurement not required. Accept $0<h<\frac{1}{3}$ or $h<\frac{1}{3}$.

No.	Evidence	Code	Judgement
2(a)(i)	For the parabola: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4a}{2y} = \frac{2a}{y}$ For the ellipse: $\frac{2x}{a^2} + \frac{2y}{b^2} \times \frac{\mathrm{d}y}{\mathrm{d}x} = 0$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-b^2x}{a^2y}$ At P(x_1,y_1) since the tangents are perpendicular and the product = –1 $\frac{2a}{y_1} \times \frac{-b^2x_1}{a^2y_1} = -1$ $2b^2x_1 = ay_1{}^2$ but (x_1,y_1) lies on $y^2 = 4ax$ (or use parameters) $2b^2x_1 = a4ax_1$ $b^2 = 2a^2$	**BS** (Possible replacement for BM)	Working with x_1 and y_1 not required. Possible alternative method using parameters is acceptable.

No.	Evidence	Code	Judgement
2(a)(ii)	At the point P the curves intersect: $4ax = b^2\left(1-\frac{x^2}{a^2}\right)$ $4a^3x = b^2\left(a^2 - x^2\right)$ $b^2x^2 + 4a^3x - a^2b^2 = 0$ but $b^2 = 2a^2$ from part (a) $2a^2x^2 + 4a^3x - 2a^4 = 0$ $x^2 + 2ax - a^2 = 0 \quad (a \neq 0)$ **OR** use $\frac{x^2}{a^2} + \frac{4ax}{2a^2} = 1$ to get the above equation $x = \frac{-2a \pm \sqrt{4a^2 + 4a^2}}{2}$ $x = -a \pm a\sqrt{2} \quad = a(-1 \pm \sqrt{2})$ **OR** use $(x+a)^2 = 2a^2$ $x + a = \pm a\sqrt{2}, \quad x = \pm a\sqrt{2} - a$ but $x \geq 0$ and since $a > 0$ $x = a(-1 + \sqrt{2})$ Hence $y^2 = 4a^2(-1+\sqrt{2})$ using the parabola and the distance OP^2 = $x^2 + y^2$ Pythagoras OP^2 = $a^2(-1+\sqrt{2})^2 + 4a^2(-1+\sqrt{2})$ $a^2(3 - 2\sqrt{2} - 4 + 4\sqrt{2})$ $= a^2(-1 + 2\sqrt{2})$ and OP = $a\sqrt{2\sqrt{2} - 1}$	**AR**	Any correct alternative method accepted (eg from perpendicular gradients). Conditions on *x* and *a* not required. Or equivalent. Accept 1.352*a* or equivalent decimal

No.	Evidence	Code	Judgement
2(b)	Using $(at^2, 2at)$ for the parabola $\frac{dy}{dx} = \frac{1}{t}$ so the tangent to the parabola is: $y - 2at = \frac{1}{t}(x - at^2)$ For M, $y = 0$ so at M $x = -at^2$ and M = $(-at^2, 0)$ Using $(a\cos\theta, b\sin\theta)$ for the ellipse: $\frac{dy}{dx} = -\frac{b\cos\theta}{a\sin\theta}$ so the tangent to the ellipse is: $y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$ For N, $y = 0$ so at N $x = \frac{a(\sin^2\theta + \cos^2\theta)}{\cos\theta}$ $x = \frac{a}{\cos\theta}$ and N = $(\frac{a}{\cos\theta}, 0)$ Hence MN = $at^2 + \frac{a}{\cos\theta}$ But at the points of intersection (P here) $at^2 = a\cos\theta = a(-1 + \sqrt{2})$ from part (a)(ii) Hence MN = $a(-1+\sqrt{2}) + \frac{a}{(-1+\sqrt{2})}$ $= a(-1+\sqrt{2}) + \frac{a(-1-\sqrt{2})}{(-1+\sqrt{2})(-1-\sqrt{2})}$ $= a(-1+\sqrt{2} + 1 + \sqrt{2})$ $= 2\sqrt{2}a$ **OR** $\text{P} = \left(a(\sqrt{2}-1), 2a\sqrt{\sqrt{2}-1}\right)$ Parabola: at P on the parabola $\frac{dy}{dx} = \frac{2a}{y} = \frac{2a}{2a\sqrt{\sqrt{2}-1}} = \frac{1}{\sqrt{\sqrt{2}-1}}$ Hence tangent is: $y - 2a\sqrt{\sqrt{2}-1} = \frac{1}{\sqrt{\sqrt{2}-1}}\left(x - a(\sqrt{2}-1)\right)$ when y=0 $-2a\left(\sqrt{2}-1\right) = x - a(\sqrt{2}-1)$ $x = -a\left(\sqrt{2}-1\right)$ at M Ellipse: at P on the ellipse	**AT** **(**Possible replacement for AP).	Any correct alternative method accepted.

	$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x}{y} = -\frac{2a(\sqrt{2}-1)}{2a\sqrt{\sqrt{2}-1}} = -\sqrt{\sqrt{2}-1}$ Hence tangent is: $y - 2a\sqrt{\sqrt{2}-1} = -\sqrt{\sqrt{2}-1}\left(x - a(\sqrt{2}-1)\right)$ when y=0 $2a = x - a(\sqrt{2}-1)$ $x = 2a + a(\sqrt{2}-1) = a(1+\sqrt{2})$ at N Hence MN = $a(1+\sqrt{2}) - \left(-a(\sqrt{2}-1)\right) = 2\sqrt{2}a$. **OR** $\mathrm{P} = \left(a(\sqrt{2}-1), 2a\sqrt{\sqrt{2}-1}\right)$ Let Q be the foot of the perpendicular from P to the *x*-axis Using gradients: $\frac{\mathrm{PQ}}{\mathrm{MQ}} = \frac{1}{\sqrt{\sqrt{2}-1}}$, but $\mathrm{PQ} = 2a\sqrt{\sqrt{2}-1}$ $\mathrm{MQ} = 2a\sqrt{\sqrt{2}-1}.\sqrt{\sqrt{2}-1} = 2a(\sqrt{2}-1)$ Similarly: $\frac{\mathrm{PQ}}{\mathrm{NQ}} = \left\lvert\frac{-2x}{y}\right\rvert = \left\lvert\frac{-2a(\sqrt{2}-1)}{2a\sqrt{\sqrt{2}-1}}\right\rvert = \sqrt{\sqrt{2}-1}$, but $\mathrm{PQ} = 2a\sqrt{\sqrt{2}-}$ $\mathrm{NQ} = \frac{2a\sqrt{\sqrt{2}-1}}{\sqrt{\sqrt{2}-1}} = 2a$ So MN=MQ+NQ= $2a(\sqrt{2}-1) + 2a = 2\sqrt{2}a$.		Accept correct use of absolute value and addition.

No.	Evidence	Code	Judgement
3(a) (i)	$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2x(1+x^2)-2x^3}{(1+x^2)^2}=\dfrac{2x}{(1+x^2)^2}$ When $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{2}$ $\dfrac{2x}{(1+x^2)^2}=\dfrac{1}{2}$ $x^4+2x^2-4x+1=0$ but x = 1 is a solution, so by division or otherwise $(x-1)(x^3+x^2+3x-1)=0$ and any other solutions are from $x^3+x^2+3x-1=0$	**BM** (Possible replacement for BS) **C**	Any correct alternative method accepted.
	Let $g(x)$ = x^3+x^2+3x-1, then $g\left(\dfrac{1}{4}\right)=\dfrac{1+4+48-64}{64}=-\dfrac{11}{64}<0$ $g\left(\dfrac{1}{2}\right)=\dfrac{1+2+12-8}{8}=\dfrac{7}{8}>0$ Hence there is a root in the interval $\left(\dfrac{1}{4},\dfrac{1}{2}\right)$ $g(x)$ = 0 for some $\dfrac{1}{4}<x<\dfrac{1}{2}$. **OR** (while the question does say HENCE, accept the following as an alternative.) $f(x)=x^4+2x^2-4x+1$ $f\left(\dfrac{1}{4}\right)=\dfrac{33}{256}=0.1289>0$ $f\left(\dfrac{1}{2}\right)=-\dfrac{7}{16}=-0.4375<0$ Hence there is a root in the interval $\left(\dfrac{1}{4},\dfrac{1}{2}\right)$ **OR** (while the question does say HENCE, accept the following as an alternative.) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2x}{(1+x^2)^2}$		accept $-\dfrac{11}{64}=-0.1719$, $\dfrac{7}{8}=0.875$ or equivalent Accept x=0.2956 obtained by Newton-Raphson method, graphic calculator, etc Explicit use of or mention of intermediate value theorem or continuity not required. Continuous

	$$x = \frac{1}{4},\ \frac{dy}{dx} = \frac{\frac{1}{2}}{(1+\frac{1}{16})^2} = \frac{128}{289} < \frac{1}{2}$$ and when $$x = \frac{1}{2},\ \frac{dy}{dx} = \frac{1}{(1+\frac{1}{4})^2} = \frac{16}{25} > \frac{1}{2}$$ note $\frac{128}{289} = 0.443,\quad \frac{16}{25} = 0.64$ or equivalent Hence there is a value of $\frac{dy}{dx} = \frac{1}{2}$ in the interval $\left(\frac{1}{4}, \frac{1}{2}\right)$ $\frac{dy}{dx} = \frac{1}{2}$ for some $\frac{1}{4} < x < \frac{1}{2}$.		derivative may be assumed. Accept decimals

No.	Evidence	Code	Judgement
3(a) (ii)	First translate the function 1 unit in the negative x direction to get $y = \frac{x^2}{1+x^2}$. The volume of revolution is then $V = \int_0^{\frac{1}{2}} \pi x^2 \mathrm{d}y$ and $y = \frac{x^2}{1+x^2}$, so rearrange to get $x^2 = \frac{y}{1-y}$ and $V = \pi \int_0^{\frac{1}{2}} \left(\frac{1}{1-y} - 1\right) \mathrm{d}y$	**BM** (Possible replacement for BS) **C**	Any correct alternative method accepted, including working with no translation Units are not required. **C** = Correct integral, but has not evaluated correctly
	$V = \pi\left[-\ln\lvert 1-y\rvert - y\right]_0^{\frac{1}{2}} = \pi(-\ln(\tfrac{1}{2}) - \tfrac{1}{2})$ $= \pi(\ln(2) - \tfrac{1}{2})$ units^3. **OR** to integrate using substitution: $V = \int_0^{\frac{1}{2}} \pi \frac{y}{1-y} \mathrm{d}y$ *let* $u = 1-y$ $\mathrm{d}u = -\mathrm{d}y$ $y = 1-u$ $V = \int_1^{\frac{1}{2}} \pi \frac{1-u}{u}(-\mathrm{d}u)$ $= \int_{\frac{1}{2}}^1 \pi\left(\frac{1}{u} - 1\right)\mathrm{d}u$ $= \pi\left[\ln\lvert u\rvert - u\right]_{\frac{1}{2}}^1$ $= \pi((0-1) - (\ln(\tfrac{1}{2}) - \tfrac{1}{2}))$ $= \pi(\ln 2 - \tfrac{1}{2})$ units^3		Final answer does not need to be simplified. Accept 0.6068, 0.193π or equivalent Units not needed

<table>
<tr><th>No.</th><th>Evidence</th><th>Code</th><th>Judgement</th></tr>
<tr><td>3(b) (i)</td><td>Consider $I = \int_0^a f(a-x)\mathrm{d}x.$

Let $u = a - x$ then $\frac{\mathrm{d}u}{\mathrm{d}x} = -1$,

and when $x = 0$, $u = a$; when $x = a$, $u = 0$, so

$I = \int_a^0 -f(u)\mathrm{d}u$
$I = \int_0^a f(u)\mathrm{d}u = \int_0^a f(x)\mathrm{d}x.$

OR
Let $F(x)$ be an antiderivative of f
$\int_0^a f(x)dx = \left[F(x)\right]_0^a = F(a) - F(0)$
$\int_0^a f(a-x)dx = \left[-F(a-x)\right]_0^a = -F(0) + F(a)$
and hence result.</td><td>AP
(Possible replacement for AT)</td><td>Any correct alternative method accepted, for example a correct transformation argument using reflection and translation.</td></tr>
<tr><td>3(b) (ii)</td><td>Using the result in 3(b)(i) with $a = \frac{\pi}{2}$

$\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x}\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{\sin^n\left(\frac{\pi}{2} - x\right)}{\sin^n\left(\frac{\pi}{2} - x\right) + \cos^n\left(\frac{\pi}{2} - x\right)}\mathrm{d}x$

$= \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\cos^n x + \sin^n x}\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x}\mathrm{d}x$

since $\sin\left(\frac{\pi}{2} - x\right) = \cos x$
and so
$\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x}\mathrm{d}x$
$= \frac{1}{2}\left\{\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x}\mathrm{d}x + \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x}\mathrm{d}x\right\}$

$= \frac{1}{2}\left\{\int_0^{\frac{\pi}{2}} \frac{\sin^n x + \cos^n x}{\sin^n x + \cos^n x}\mathrm{d}x\right\} = \frac{1}{2}\left\{\frac{\pi}{2}\right\} = \frac{\pi}{4}$</td><td>AT</td><td>No alternative.</td></tr>
</table>

No.	Evidence	Code	Judgement
4(a) (i)	$\lvert v\rvert = \sqrt{2}$ Using de Moivre's Theorem: $\lvert vz\rvert = \lvert v\rvert\lvert z\rvert = \sqrt{2}\lvert z\rvert$ So $\lvert z-v\rvert = \lvert vz\rvert$ becomes $\lvert z-v\rvert = \sqrt{2}\lvert z\rvert$ Squaring $\lvert z-v\rvert^2 = 2\lvert z\rvert^2$ Letting *z* = *x* +i*y* we get $(x-1)^2+(y-1)^2 = 2(x^2+y^2)$	BS C	
	and $x^2+2x+y^2+2y=2$ $(x+1)^2+(y+1)^2=4$ a circle centre (–1, –1) and radius 2. **OR** use $vz=(1+i)(x+iy)=(x-y)+i(x+y)$ So $\lvert z-v\rvert = \lvert vz\rvert$ becomes $(x-1)^2+(y-1)^2=(x-y)^2+(x+y)^2$	 OR C	Either form of the circle equation will be accepted. No alternative.
	$-2x-2y+2=x^2+y^2$ and as above $x^2+2x+y^2+2y=2$ $(x+1)^2+(y+1)^2=4$ a circle centre (–1, –1) and radius 2.		
4(a) (ii)	The line $\lvert z-v\rvert = \lvert z+v\rvert$ means that *z* lies on the perpendicular bisector of the line joining (1, 1) and (–1, –1) in the complex plane, ie $y=-x$. **OR** $\lvert z-v\rvert = \lvert z+v\rvert$ so $(x-1)^2+(y-1)^2=(x+1)^2+(y+1)^2$ and $-2x-2y=2x+2y,\quad y=-x.$	BS OR C	Any correct alternative method accepted. Accept ONE **C** from either (a)(i) OR (a)(ii)
	This meets the circle $(x+1)^2+(y+1)^2=4$ where $(x+1)^2+(-x+1)^2=4,\quad 2x^2=2$ $x^2=1,\quad x=\pm1$ and $y=\mp1$ ie at the points (1, –1) and (–1, 1).		Accept $z=\pm1\mp i$

No.	Evidence	Code	Judgement
4(b)	$z^5 - 1 = 0,$ $(z-1)(z^4 + z^3 + z^2 + z + 1) = 0$ roots of $z^5 - 1 = 0,$ are $z = \text{cis}\left(\frac{2k\pi}{5}\right),\ k = 0,\ 1,\ 2,\ 3,\ 4$ and k = 0 gives z = 1, so $z^4 + z^3 + z^2 + z + 1 = \left(z - \text{cis}\left(\frac{2\pi}{5}\right)\right)\left(z - \text{cis}\left(\frac{4\pi}{5}\right)\right)$ $\left(z - \text{cis}\left(\frac{6\pi}{5}\right)\right)\left(z - \text{cis}\left(\frac{8\pi}{5}\right)\right)$ $= \left(z - \text{cis}\left(\frac{2\pi}{5}\right)\right)\left(z - \text{cis}\left(\frac{4\pi}{5}\right)\right)$ $\left(z - \text{cis}\left(-\frac{4\pi}{5}\right)\right)\left(z - \text{cis}\left(-\frac{2\pi}{5}\right)\right)$ $\left(z^2 - \left(\text{cis}\left(\frac{2\pi}{5}\right) + \text{cis}\left(-\frac{2\pi}{5}\right)\right)z + \text{cis}\left(\frac{2\pi}{5}\right)\text{cis}\left(-\frac{2\pi}{5}\right)\right)$ = $\left(z^2 - \left(\text{cis}\left(\frac{4\pi}{5}\right) + \text{cis}\left(-\frac{4\pi}{5}\right)\right)z + \text{cis}\left(\frac{4\pi}{5}\right)\text{cis}\left(-\frac{4\pi}{5}\right)\right)$ but $\text{cis}\left(\frac{2\pi}{5}\right) + \text{cis}\left(-\frac{2\pi}{5}\right) = 2\cos\left(\frac{2\pi}{5}\right)$ and $\text{cis}\left(\frac{4\pi}{5}\right) + \text{cis}\left(-\frac{4\pi}{5}\right) = 2\cos\left(\frac{4\pi}{5}\right)$ also $\text{cis}\left(\frac{2\pi}{5}\right)\text{cis}\left(-\frac{2\pi}{5}\right) = \text{cis}(0) = 1$ and $\text{cis}\left(\frac{4\pi}{5}\right)\text{cis}\left(-\frac{4\pi}{5}\right) = 1$	**AR**	Any correct alternative method accepted. Factorisation required. Accept $(z^2 - 0.618z + 1)(z^2 + 1.618z + 1)$

	so $z^4 + z^3 + z^2 + z + 1 =$ $\left(z^2 - 2\cos\left(\frac{2\pi}{5}\right)z + 1\right)\left(z^2 - 2\cos\left(\frac{4\pi}{5}\right)z + 1\right)$ Comparing coefficients of z^2 1 = $1 + 1 + 4\cos\left(\frac{2\pi}{5}\right)\cos\left(\frac{4\pi}{5}\right)$ and $\cos\left(\frac{2\pi}{5}\right)\cos\left(\frac{4\pi}{5}\right) = -\frac{1}{4}$		or $\left(z^2 - \left(\frac{-1+\sqrt{5}}{2}\right)z + 1\right)\left(z^2 - \left(\frac{-1-\sqrt{5}}{2}\right)z + 1\right)$ for S1 No alternative answer.

No.	Evidence	Code	Judgement
5(a)	$x = r\cos\theta$ $\quad y = r\sin\theta$ $= \mathrm{A}e^{k\theta}\cos\theta$ $\quad = \mathrm{A}e^{k\theta}\sin\theta$ $\frac{dx}{d\theta} = \cos\theta \mathrm{A}ke^{k\theta} - \sin\theta.\mathrm{A}e^{k\theta}$ $= \mathrm{A}e^{k\theta}(k\cos\theta - \sin\theta)$ $\frac{dy}{d\theta} = \mathrm{A}ke^{k\theta}\sin\theta + \mathrm{A}e^{k\theta}\cos\theta$ $= \mathrm{A}e^{k\theta}(k\sin\theta + \cos\theta)$ $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{\mathrm{A}e^{k\theta}(k\sin\theta + \cos\theta)}{\mathrm{A}e^{k\theta}(k\cos\theta - \sin\theta)}$ $= \frac{k\sin\theta + \cos\theta}{k\cos\theta - \sin\theta}$ $= \frac{k\tan\theta + 1}{k - \tan\theta}$	**BM**	Either form of $\frac{dy}{dx}$, or equivalent, will be accepted.

No.	Evidence	Code	Judgement
5(b)	$\frac{dy}{dx} = \tan(\alpha + \theta)$ (gradient of the tangent using exterior angle of a triangle) (see diagram) so $\frac{\tan\alpha + \tan\theta}{1 - \tan\alpha\tan\theta} = \frac{k\tan\theta + 1}{k - \tan\theta}$ $\frac{\tan\alpha + \tan\theta}{1 - \tan\alpha\tan\theta} = \frac{\tan\theta + \frac{1}{k}}{1 - \frac{1}{k}\tan\theta}$ and $\tan\alpha = \frac{1}{k}$ $\alpha = \tan^{-1}\left(\frac{1}{k}\right)$ **OR** use $\left(\tan\alpha + \tan\theta\right)\left(k - \tan\theta\right) = \left(1 - \tan\alpha\tan\theta\right)\left(k\tan\theta + 1\right)$ $\left(k\tan\alpha - 1\right)\left(\tan^2\theta + 1\right) = 0$ $\tan\alpha = \frac{1}{k}$ since $\tan^2\theta + 1 \neq 0$ $\alpha = \tan^{-1}\left(\frac{1}{k}\right)$ **OR** $\frac{dy}{dx} = \tan(\alpha + \theta)$ and from (a) $\frac{dy}{dx} = \frac{k\sin\theta + \cos\theta}{k\cos\theta - \sin\theta}$ hence $\tan(\alpha + \theta) = \frac{k\sin\theta + \cos\theta}{k\cos\theta - \sin\theta}$ and this is true for all values of θ. In particular it is true for θ=0, and then $\tan(\alpha + 0) = \frac{k\sin 0 + \cos 0}{k\cos 0 - \sin 0} = \frac{1}{k}$ so $\tan\alpha = \frac{1}{k}$ and $\alpha = \tan^{-1}\left(\frac{1}{k}\right)$.	**AP**	Accept $\tan\alpha = \frac{1}{k}$, or equivalent. Accept $\frac{\pi}{2} - \tan^{-1}(k)$, or equivalent.

No.	Evidence	Code	Judgement
6(a)	$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=0$, integrating wrt t $\quad \frac{\mathrm{d}x}{\mathrm{d}t}=C$ but when t = 0, v_x = $V\cos\alpha$, so $\frac{\mathrm{d}x}{\mathrm{d}t}=V\cos\alpha$ $\frac{\mathrm{d}^2y}{\mathrm{d}t^2}=-\mathrm{g}$, integrating wrt t, $\frac{\mathrm{d}y}{\mathrm{d}t}=-\mathrm{g}t+K$ but when t = 0, v_y = $V\sin\alpha$, so K = $V\sin\alpha$ and $\frac{\mathrm{d}y}{\mathrm{d}t}=-\mathrm{g}t+V\sin\alpha$	**BM** **(**Possible replacement for BS) **C**	
	Integrating again $\frac{\mathrm{d}x}{\mathrm{d}t}=V\cos\alpha$ so $x=Vt\cos\alpha+M$ but x = 0 when t = 0, so M = 0 and $x=Vt\cos\alpha$ $\frac{\mathrm{d}y}{\mathrm{d}t}=-\mathrm{g}t+V\sin\alpha$ so $y=\frac{-\mathrm{g}t^2}{2}+Vt\sin\alpha+N$ but y = 0 when t = 0, so N = 0 and $y=\frac{-\mathrm{g}t^2}{2}+Vt\sin\alpha$ From $x=Vt\cos\alpha$ we get $t=\frac{x}{V\cos\alpha}$ and substituting into $y=\frac{-\mathrm{g}t^2}{2}+Vt\sin\alpha$ gives $y=\frac{-\mathrm{g}\left(\frac{x}{V\cos\alpha}\right)^2}{2}+V\left(\frac{x}{V\cos\alpha}\right)\sin\alpha$ so $y=x\tan\alpha-\frac{\mathrm{g}x^2}{2V^2}\sec^2\alpha$ or $y=x\tan\alpha-\frac{\mathrm{g}x^2}{2V^2}(1+\tan^2\alpha)$		Accept parametric answer: $x=Vt\cos\alpha$ $y=\frac{-\mathrm{g}t^2}{2}+Vt\sin\alpha$ this form only

<table>
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<tr><td>6(b) (i)</td><td>If the centre passes through the point (kh, h),
then from the above
$h = kh\tan\alpha - \frac{g(kh)^2}{2V^2}(1+\tan^2\alpha)$
which is a quadratic in tan α, so rearranging
$g(kh)^2\tan^2\alpha - 2V^2kh\tan\alpha + 2V^2h + g(kh)^2 = 0$
and, $h \neq 0$

$ghk^2\tan^2\alpha - 2V^2k\tan\alpha + 2V^2 + ghk^2 = 0$

This has two distinct real solutions
if and only if $b^2 - 4ac > 0$

$\left(2V^2k\right)^2 - 4ghk^2(2V^2 + ghk^2) > 0$

$4V^4k^2 - 4ghk^2(2V^2 + ghk^2) > 0$

$V^4 - 2ghV^2 - g^2h^2k^2 > 0 \quad (k^2 > 0)$

the LHS of which is a quadratic in V^2, and has the form $(V^2 - a)(V^2 - b)$ where a, b arise from

$V^2 = \frac{2gh \pm \sqrt{(2gh)^2 + 4g^2h^2k^2}}{2}$

$V^2 = \frac{gh \pm gh\sqrt{1+k^2}}{1}$

$V^2 = gh\left(1 \pm \sqrt{1+k^2}\right)$ but $1 - \sqrt{1+k^2} < 0$

so $4V^4 - 8ghV^2 - 4g^2k^2h^2 > 0$ when
$V^2 > gh\left(1+\sqrt{1+k^2}\right)$
OR
$y = \frac{-gt^2}{2} + Vt\sin\alpha$ and $x = Vt\cos\alpha$
since it passes through the point (kh, h)
$h = \frac{-gt^2}{2} + Vt\sin\alpha$ and $kh = Vt\cos\alpha$
$Vt\sin\alpha = h + \frac{gt^2}{2}$
$V^2t^2\sin^2\alpha = \left(h + \frac{gt^2}{2}\right)^2$</td><td>AT
(Possible replacement for AR).</td><td></td></tr>
</table>

	$V^2t^2\left(1-\left(\frac{kh}{Vt}\right)^2\right)=\left(h+\frac{gt^2}{2}\right)^2$ $4\left(V^2t^2-k^2h^2\right)=4h^2+4ght^2+g^2t^4$ $g^2t^4+4\left(gh-V^2\right)t^2+4h^2\left(1+k^2\right)=0$ and to get 2 different values of t we need $b^2-4ac>0$ $16\left(gh-V^2\right)^2-16g^2h^2\left(1+k^2\right)>0$ $\left(gh-V^2\right)^2-g^2h^2\left(1+k^2\right)>0$ $gh-V^2<-gh\sqrt{1+k^2}$ or $gh-V^2>gh\sqrt{1+k^2}$ $V^2>\mathrm{g}h\left(1+\sqrt{1+k^2}\right)$ ($V^2<gh\left(1-\sqrt{1+k^2}\right)$ not possible since $1-\sqrt{1+k^2}<0$).		

No.	Evidence	Code	Judgement
6(b) (ii)	Use $g(kh)^2\tan^2\alpha - 2V^2kh\tan\alpha + 2V^2h + g(kh)^2 = 0$ Let $\tan\alpha_1$, $\tan\alpha_2$ be the two roots, then by the sum and the product of the roots: $\tan\alpha_1 + \tan\alpha_2 = \dfrac{2V^2kh}{gk^2h^2} = \dfrac{2V^2}{gkh}$ and $\tan\alpha_1 \times \tan\alpha_2 = \dfrac{2V^2h + gk^2h^2}{gk^2h^2} = \dfrac{2V^2 + gk^2h}{gk^2h}$ and, since $\tan(\alpha_1+\alpha_2) = \dfrac{\tan\alpha_1 + \tan\alpha_2}{1-\tan\alpha_1 \times \tan\alpha_2}$ $\tan(\alpha_1+\alpha_2) = \dfrac{\frac{2V^2}{gkh}}{1-\frac{2V^2+gk^2h}{gk^2h}}$ $= \dfrac{2kV^2}{gk^2h-2V^2-gk^2h}$ $= -k$ and $\alpha_1+\alpha_2 = \tan^{-1}(-k)$.	**AP** (Possible replacement for AR or AT).	